

Winding vacuum energies in a deformed $O(4)$ sigma model

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Abstract

We consider the problem of calculating the Casimir energies in the winding sectors of Fateev's SS -model, which is an integrable two-parameter deformation of the $O(4)$ non-linear sigma model in two dimensions. This problem lies beyond the scope of all traditional methods of integrable quantum field theory including the thermodynamic Bethe ansatz and non-linear integral equations. Here we propose a solution based on a remarkable correspondence between classical and quantum integrable systems and express the winding energies in terms of certain solutions of the classical sinh-Gordon equation.

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1. Introduction

Non-Linear Sigma Models (NLSM) are perhaps the most interesting among two-dimensional (2D) models of quantum field theory, both in terms of the mathematical physics involved, and especially in terms of their applications. As a classical field theory, NLSM in the simplest setup describe harmonic maps from a 2D surface to a Riemannian manifold (the target space).

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In physics NLSM were first introduced in the beginning of the 1960s as four-dimensional phenomenological models describing the effective interaction of mesons in the chiral limit. The interest to 2D NLSM was inspired by Polyakov in 1975 [1] who pointed out that the $O(N)$ -sigma model (whose target space is a round $(N - 1)$ -sphere) is an asymptotically free theory and, thus, could serve as an ideal laboratory for studying the four-dimensional Yang–Mills theories.

The case of the $O(4)$ -sigma model is somewhat special. The round 3-sphere possesses a group structure so that the model is equivalent to the $SU(2)$ principal chiral field:

$$\mathcal{A}_{PC} = \frac{1}{4u} \int d^2x \operatorname{Tr}(\partial_\mu \mathbf{g} \partial^\mu \mathbf{g}^{-1}) \quad (\mathbf{g} \in SU(2)), \quad (1.1)$$

where the coupling constant coincides with the inverse square of the radius — the only metric parameter of the round sphere. In the work [2] it was discovered that the following two parameter deformation of the principal chiral field

$$\mathcal{A}_{SS} = \int d^2x \frac{u \operatorname{Tr}(\partial_\mu \mathbf{g} \partial^\mu \mathbf{g}^{-1}) + 2l(L_\mu^3)^2 + 2r(R_\mu^3)^2}{4(u+r)(u+l) - rl(\operatorname{Tr}(\mathbf{g}\sigma_3\mathbf{g}^{-1}\sigma_3))^2} \quad (1.2)$$

is a renormalizable NLSM within a three-dimensional space of couplings (u, r, l) at the one-loop level (here L_μ^3 and R_μ^3 stand for the left and right currents: $L_\mu^3 := \frac{1}{2i} \operatorname{Tr}(\partial_\mu \mathbf{g} \mathbf{g}^{-1} \sigma_3)$, $R_\mu^3 := \frac{1}{2i} \operatorname{Tr}(\mathbf{g}^{-1} \partial_\mu \mathbf{g} \sigma_3)$). The following combinations of parameters turned out to be renormalization group (RG) invariant:

$$a_1 = +\frac{\pi}{2\sqrt{l(u+r)}}, \quad a_2 = +\frac{\pi}{2\sqrt{r(u+l)}}. \quad (1.3)$$

Moreover, Fateev presented a set of convincing arguments in favor of the quantum integrability of the model (1.2). In particular, he argued that its spectrum is generated by two massive doublets of the same mass whose 2-particle S -matrix has the form of a direct product $(-S_{a_1} \otimes S_{a_2})$ of two $U(1)$ -symmetric solutions of the S -matrix bootstrap equations. For this reason the above two-parameter deformation of the $O(4)$ -sigma model was named the SS -model. Also, it is worth noting, that S_a coincides with the soliton S -matrix [3] in the quantum sine-Gordon theory with the renormalized coupling constant a .

In this work we impose the twisted boundary condition for the matrix valued field \mathbf{g} ,

$$\mathbf{g}(t, x + R) = e^{i\pi k_2 \sigma_3} \mathbf{g}(t, x) e^{i\pi k_1 \sigma_3}. \quad (1.4)$$

The space of states of the theory then splits into sectors characterized by a pair of “winding” numbers, $\mathbf{k} = (k_1, k_2)$. The ground-state in each sector is referred to below as the k -vacuum and the corresponding energy is denoted by $E_{\mathbf{k}}^{(\text{vac})}$.

The lowest vacuum energy $E_{\mathbf{k}=0}^{(\text{vac})}$ can be calculated in the framework of the Thermodynamic Bethe Ansatz (TBA) approach. For the simplest case of integer parameters $a_1, a_2 = 2, 3, 4, \dots$, the required TBA equations were obtained in [2]. These equations are encoded by the incidence diagram shown in Fig. 1, which has one massive node.¹ Subsequently, in Ref. [5], these equations were generalized to a system of Non-Linear Integral Equations (NLIE) [6,7] which allows

¹ As noted in [4], if a model has an S -matrix in the form of a direct product $(-S_G \otimes S_H)$ and the TBA equations for the models described by S -matrices S_G and S_H are encoded by Dynkin-like diagrams of type G and H , each having one massive node, then the TBA equations for the model with the direct product S -matrix are obtained by “gluing” together the individual TBA equations at their massive nodes. This prescription, when applied to the SS -model with integer $a_1, a_2 \geq 2$, leads to a TBA system whose incidence diagram is shown in Fig. 1.

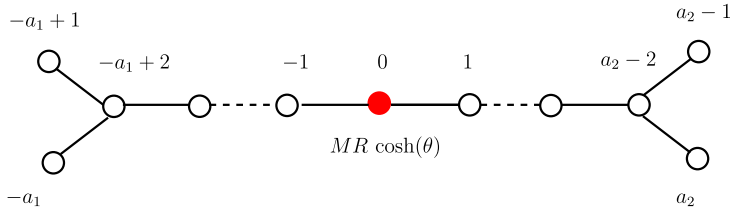


Fig. 1. Incidence diagram for the TBA system describing the vacuum energy at the sector $k_1 = k_2 = 0$ in the case $a_1, a_2 = 2, 3, 4, \dots$. The source term is indicated near the corresponding node.

one to calculate $E_{\mathbf{k}=0}^{(\text{vac})}$ for any values of $a_1, a_2 \geq 2$. Moreover, the $\mathbf{k} = 0$ case of the undeformed $O(4)$ -sigma model was separately considered in Refs. [8–10]. However, to the best of our knowledge, the problem of calculating the k -vacuum energies for general values of a_i and k_i is beyond the scope of traditional approaches of integrable quantum field theory. The purpose of this note is to extend a result of earlier work [11] and conjecture an exact formula for the k -vacuum energy in the SS -model in the general case.

2. UV/IR behavior of k -vacuum energy

Although $E_{\mathbf{k}}^{(\text{vac})}$ is a rather complicated function of the parameters, its leading small- R (i.e., UV) and large- R (IR) behavior can be obtained via a simple and intuitive analysis which is based on the dual form of the SS -model proposed in Ref. [2].

The dual description is formulated in terms of three Bose fields governed by the Toda-like Lagrangian

$$\tilde{\mathcal{L}}_{SS} = \frac{1}{16\pi} \sum_{i=1}^3 (\partial_\mu \varphi_i \partial^\mu \varphi_i)^2 + 2\mu (e^{b\varphi_3} \cos(\alpha_1 \varphi_1 + \alpha_2 \varphi_2) + e^{-b\varphi_3} \cos(\alpha_1 \varphi_1 - \alpha_2 \varphi_2)), \quad (2.1)$$

where

$$\alpha_i = \frac{1}{2} \sqrt{a_i}, \quad b = \frac{1}{2} \sqrt{a_1 + a_2 - 2} \quad (2.2)$$

and the dimensionfull coupling μ is related to the soliton mass as

$$M = 2\mu \frac{\Gamma(2\alpha_1^2) \Gamma(2\alpha_2^2)}{\Gamma(2\alpha_1^2 + 2\alpha_2^2)}. \quad (2.3)$$

The soliton charges $q_i = 0, \pm 1, \pm 2, \dots$, corresponding to the factors S_{a_i} ($i = 1, 2$) in the direct product $(-S_{a_1} \otimes S_{a_2})$, appear through the quasiperiodic boundary conditions imposed on the dual fields:

$$\varphi_1(x_1 + R) = \varphi_1(x_1) + \frac{\pi}{\alpha_1} (q_2 + q_1), \quad \varphi_2(x_1 + R) = \varphi_2(x_1) + \frac{\pi}{\alpha_2} (q_2 - q_1). \quad (2.4)$$

In their turn, the winding numbers are interpreted as quasimomenta. Due to the periodicity of the potential terms in φ_j ($j = 1, 2$), the stationary states can be chosen to be the Floquet states characterized by the pair $\mathbf{k} = (k_1, k_2)$:

$$\varphi_i \mapsto \varphi_i + 2\pi/\alpha_i: \quad |\Psi_{\mathbf{k}}\rangle \mapsto e^{2\pi i k_i} |\Psi_{\mathbf{k}}\rangle. \quad (2.5)$$

The form of the dual Lagrangian suggests that for small R

$$E_{\mathbf{k}}^{(\text{vac})} \approx \frac{\pi}{R} \left(-\frac{1}{2} + \frac{p_0^2}{4b^2} + a_1 k_1^2 + a_2 k_2^2 \right). \quad (2.6)$$

Since values of the field φ_3 is effectively restricted within the segment of length $(-2b \log(\mu R))$, the corresponding “zero-mode momentum” p_0 is not arbitrary. It is determined through a certain quantization condition, similar to that discussed in Ref. [12] in the context of the quantum sinh-Gordon model. Assuming that

$$|a_1 k_1 \pm a_2 k_2| < 1, \quad (2.7)$$

the original consideration from [12] can be applied to the SS-model yielding

$$-\frac{p_0}{b^2} \log\left(\frac{\mu R}{8b^2}\right) + \delta^{(\mathbf{q}=0)}(p_0) \approx 2\pi, \quad (2.8)$$

with

$$\delta^{(\mathbf{q})}(p) = -i \log(S^{(q_1)}(p|a_1 k_1 - a_2 k_2) S^{(q_2)}(p|a_1 k_1 + a_2 k_2)) \quad (\delta^{(\mathbf{q})}(0) = 0). \quad (2.9)$$

Here $S^{(\mathbf{q})}(p|\lambda)$ stands for the so-called “reflection amplitude” for the sine-Liouville model [13]

$$S^{(q)}(p|\lambda) = \frac{\Gamma(\frac{1+|q|}{2} + \frac{\lambda}{2} - \frac{ip}{2}) \Gamma(\frac{1+|q|}{2} - \frac{\lambda}{2} - \frac{ip}{2})}{\Gamma(\frac{1+|q|}{2} + \frac{\lambda}{2} + \frac{ip}{2}) \Gamma(\frac{1+|q|}{2} - \frac{\lambda}{2} + \frac{ip}{2})} \frac{\Gamma(1+ip) \Gamma(1 + \frac{ip}{4b^2})}{\Gamma(1-ip) \Gamma(1 - \frac{ip}{4b^2})}. \quad (2.10)$$

In the IR limit the k -vacuum energy is composed of an extensive part proportional to the length of the system

$$E_{\mathbf{k}}^{(\text{vac})} = R\mathcal{E}_0 + o(1) \quad \text{as } R \rightarrow \infty. \quad (2.11)$$

The exact form of the specific bulk energy was found in [2]. It is expressed through the soliton mass M as

$$\mathcal{E}_0 = -\frac{M^2}{4} \frac{\sin(\frac{\pi}{2}a_1) \sin(\frac{\pi}{2}a_2)}{\sin(\frac{\pi}{2}(a_1 + a_2))}. \quad (2.12)$$

In the case $a_1, a_2 > 1$, when the fundamental particles do not form bound states, the leading correction to (2.11) comes from virtual soliton and antisoliton trajectories winding once around the space circle. These trajectories should be counted with the phase factor $e^{i\pi(\sigma_1 k_1 + \sigma_2 k_2)}$, where $\sigma_{1,2} = \pm 1$. Therefore, summing over the four possible sign combinations one obtains

$$E_{\mathbf{k}}^{(\text{vac})} = R\mathcal{E}_0 - \frac{4}{\pi} \cos(\pi k_1) \cos(\pi k_2) M K_1(MR) + (\text{multiparticle}) \quad (a_{1,2} > 1) \quad (2.13)$$

(here $K_1(r)$ stands for the conventional Bessel function). Note that similar arguments were originally applied to the quantum sine-Gordon model by Al. Zamolodchikov in Ref. [14].

In Fig. 2 the UV/IR asymptotic formulae are compared with the results of a numerical solution of the TBA system described by the incidence diagram from Fig. 1.

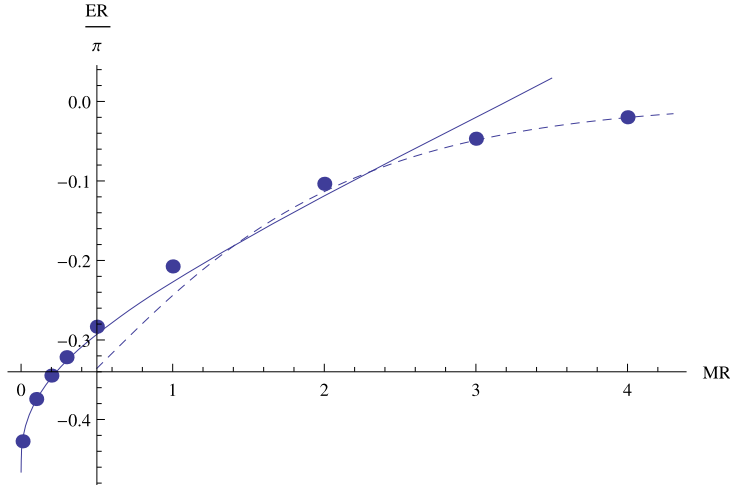


Fig. 2. Numerical values of the dimensionless k -vacuum energy $\frac{R}{\pi} E_{\mathbf{k}}^{(\text{vac})}$ versus MR for $a_1 = 2$, $a_2 = 3$ and $k_1 = k_2 = 0$. The solid and dashed lines follow from UV (2.6), and IR (2.13) asymptotic formulas, respectively. The heavy dots were obtained by means of a numerical solution of the TBA system encoded by the incidence diagram of Fig. 1.

3. Exact k -vacuum energy

3.1. Fateev model

The model governed by the Lagrangian

$$\mathcal{L}_F = \frac{1}{16\pi} \sum_{i=1}^3 (\partial_\mu \varphi_i \partial^\mu \varphi_i)^2 + 2\mu (e^{i\alpha_3 \varphi_3} \cos(\alpha_1 \varphi_1 + \alpha_2 \varphi_2) + e^{-i\alpha_3 \varphi_3} \cos(\alpha_1 \varphi_1 - \alpha_2 \varphi_2)), \quad (3.1)$$

where the coupling constants α_i are subjected to a single constraint

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \frac{1}{2}, \quad (3.2)$$

will be referred to below as the *Fateev model*. In the case when α_1, α_2 are real while α_3 is pure imaginary (*unitary regime*), the Lagrangian (3.1) is real and coincides with the dual Lagrangian $\tilde{\mathcal{L}}_{SS}$ provided $\alpha_3 = -ib$. In the *symmetric regime* all the coupling constant α_i are real, the Lagrangian (3.1) is completely symmetric under simultaneous permutations of the real fields φ_i and couplings α_i . Despite that the theory is apparently non-unitary in this case, one can still address the problem of calculation of the k -vacuum energies. Since the Lagrangian \mathcal{L}_F in the symmetric regime is invariant under the transformations $\varphi_i \mapsto \varphi_i + 2\pi\alpha_i$ with $i = 1, 2, 3$, the k -vacuum energies are labeled by the triple of quasimomenta $\mathbf{k} = (k_1, k_2, k_3)$ (contrary to the unitary regime where $\mathbf{k} = (k_1, k_2)$). The short distance expansion of $E_{\mathbf{k}}^{(\text{vac})}$ in the symmetric regime is considerably simpler than in the unitary one. Its general structure follows from the fact that the potential term of \mathcal{L}_F with $\alpha_i > 0$ is a uniformly bounded perturbation for any values of the dimensionless

parameter μR . Therefore the conformal perturbation theory can be applied literally yielding an expansion

$$\text{Symmetric regime: } \frac{R}{\pi} E_{\mathbf{k}}^{(\text{vac})} = -\frac{1}{2} + \sum_{i=1}^3 (2\alpha_i k_i)^2 - \sum_{n=1}^{\infty} e_n (\mu R)^{4n}. \quad (3.3)$$

An exact formula for the k -vacuum energies in the symmetric regime was proposed in Ref. [11]. Below we argue that essentially the same formula actually holds in both regimes of the Fateev model.

3.2. Regular solutions of the shG equation

Consider the classical partial differential equation

$$\partial_z \partial_{\bar{z}} \hat{\eta} - \rho^2 |\mathcal{P}(z)| (e^{2\hat{\eta}} - e^{-2\hat{\eta}}) = 0 \quad (3.4)$$

where

$$\mathcal{P}(z) = \frac{(z_3 - z_2)^{a_1} (z_1 - z_3)^{a_2} (z_2 - z_1)^{a_3}}{(z - z_1)^{2-a_1} (z - z_2)^{2-a_2} (z - z_3)^{2-a_3}} \quad (3.5)$$

and \bar{z} denotes the complex conjugate of z . Here ρ is a real parameter and a_i ($i = 1, 2, 3$) are also real and satisfy the condition

$$a_1 + a_2 + a_3 = 2. \quad (3.6)$$

The variable z is regarded as a complex coordinate on $\mathbb{CP}^1 \setminus \{z_1, z_2, z_3\}$, the Riemann sphere with three punctures. Due to the relation (3.6), $\mathcal{P}(z)(dz)^2$ is a quadratic differential under $\text{PSL}(2, \mathbb{C})$ transformations, so that the punctures can be sent to any prescribed positions, say $(z_1, z_2, z_3) = (0, 1, \infty)$. Then the change of variables

$$w = \rho \int dz z^{\frac{a_1}{2}-1} (1-z)^{\frac{a_2}{2}-1} \quad (3.7)$$

brings (3.4) to the standard form of the sinh-Gordon (shG) equation,

$$\partial_w \partial_{\bar{w}} \hat{\eta} - e^{2\hat{\eta}} + e^{-2\hat{\eta}} = 0. \quad (3.8)$$

In the case when a_1, a_2, a_3 are all positive Eq. (3.7) defines the Schwarz–Christoffel mapping, transforming the upper and lower half-planes correspondingly to the triangles (w_1, w_2, w_3) and (w_1, w_2, \bar{w}_3) , depicted in Fig. 3a. Note, that the adjacent sides of the resulting polygon $(w_1, w_2, w_3, \bar{w}_3)$ should be identified to form a topological 2-sphere. In the case when $a_3 < 0$, but $a_1, a_2 > 0$, the image of the punctured sphere is shown in Fig. 3b. Again, the adjacent rays should be properly identified. In this way Eq. (3.4) on the thrice-punctured sphere can be equivalently formulated as the shG equation in the domains shown in Fig. 3a and Fig. 3b, corresponding to the two cases

$$\begin{aligned} \text{Regime I: } & a_1 > 0, \quad a_2 > 0, \quad a_3 = 2 - a_1 - a_2 > 0 \\ \text{Regime II: } & a_1 > 0, \quad a_2 > 0, \quad a_3 = 2 - a_1 - a_2 < 0. \end{aligned} \quad (3.9)$$

We will consider *regular* solutions to (3.8), defined by the following two requirements. First, the regular solution should be a *smooth, single valued, real* function on the punctured sphere $\mathbb{CP}^1 \setminus \{z_1, z_2, z_3\}$ or, equivalently (when the complex coordinate w is employed) in the domains

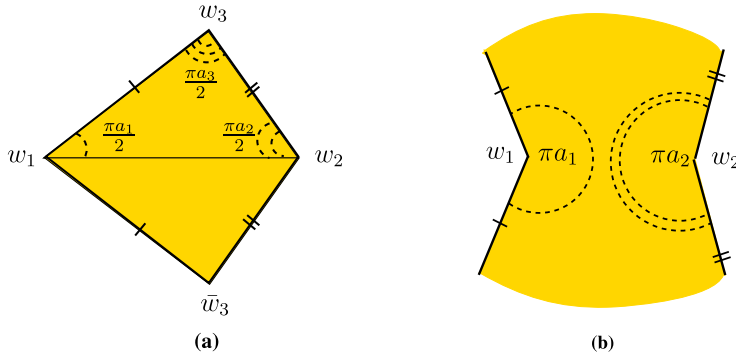


Fig. 3. The image of the thrice-punctured sphere in the complex w -plane: (a) for the case $a_{1,2,3} > 0$ (regime I); (b) for the case $a_{1,2} > 0$ and $a_3 < 0$ (regime II).

shown in Fig. 3 with properly identified edges. Second, the regular solution must develop the proper asymptotic behavior in the vicinity of the punctures. For regime I there is the freedom to control the asymptotic behavior of $\hat{\eta}$ at each of the three punctures, or, equivalently, at each vertex w_i in Fig. 3a. Namely,

$$\hat{\eta} \rightarrow 2l_i \log |w - w_i| + O(1), \quad \text{when } w \rightarrow w_i, \quad (3.10)$$

where

$$-\frac{1}{2} < l_i \leq 0 \quad (3.11)$$

denote free parameters.² For regime II, when $a_3 < 0$, the third puncture is mapped to the infinity of the domain, shown in Fig. 3b, and we require that

$$\text{Regime II: } \hat{\eta} \rightarrow 0 \quad \text{as } |w| \rightarrow \infty, \quad (3.12)$$

whereas the asymptotic behavior in the vicinity of $w = w_1, w_2$, is still described by (3.10) with two free parameters (3.11). It turns out that the solution of the shG equation, satisfying the above regularity conditions, exists and is unique for both regimes I and II.

3.3. Main conjecture

Define the functional

$$\mathfrak{F}(\rho) = -\frac{8}{\pi} \int d^2 w \sinh^2(\hat{\eta}) + \sum_i a_i l_i^2, \quad (3.13)$$

² For $l_i = -\frac{1}{2}$ the leading asymptotics (3.10) should be replaced by

$$\hat{\eta} \rightarrow -\log \left(|w - w_i| \log \left(\frac{4}{|w - w_i|} \right) \right) + O(1).$$

where $\hat{\eta}$ is a regular solution and the summation index i takes the values $i = 1, 2, 3$ and $i = 1, 2$ for the regimes I and II, respectively. The additive constant in (3.13) is chosen to provide the normalization condition

$$\lim_{\rho \rightarrow \infty} \mathfrak{F}(\rho) = 0. \quad (3.14)$$

Now we can extend the conjecture of Ref. [11] and propose the expression for the k -vacuum energies, which is valid for both considered regimes,

$$\frac{R}{\pi} E_{\mathbf{k}}^{(\text{vac})} = \mathfrak{F}(\rho) - 4\rho^2 \prod_{i=1}^3 \gamma\left(\frac{a_i}{2}\right), \quad (3.15)$$

where $\gamma(x) := \frac{\Gamma(x)}{\Gamma(1-x)}$. This formula should be supplemented with the relations between the parameters of quantum and classical problems:

$$\mu R = 2\rho, \quad \alpha_i^2 = \frac{a_i}{4}, \quad |k_i| = l_i + \frac{1}{2}. \quad (3.16)$$

In the case of the symmetric regime, formula (3.15) can be checked, in principle, perturbatively. Namely, let us return to the original variable z and replace $\hat{\eta}$ by $\eta = \hat{\eta} + \frac{1}{2} \log |\rho^2 \mathcal{P}|$. This brings (3.4) to the form of the modified shG equation:

$$\partial_z \partial_{\bar{z}} \eta - e^{2\eta} + \rho^4 |\mathcal{P}|^2 e^{-2\eta} = 0. \quad (3.17)$$

For the regular solution the third term in (3.17) can be treated perturbatively even in the nearest neighbor of each puncture and the RHS of (3.15) admits a Taylor expansion (see [11] for details):

$$\text{Regime I: } \mathfrak{F} - 4\rho^2 \prod_{i=1}^3 \gamma\left(\frac{a_i}{2}\right) = \sum_{n=0}^{\infty} f_n \rho^{4n}. \quad (3.18)$$

On the other hand, the LHS of (3.15) possesses a series expansion (3.3) which, in principle, can be obtained using the conformal perturbation theory. Thus, in the symmetric regime (regime I), both sides of (3.15) can be understood perturbatively and the conjectured relation implies that the corresponding expansion coefficients are simply related: $f_n = -2^{4n} e_n$.

The situation is somewhat different in the unitary regime (regime II). Of course, the RHS of (3.15) in this regime is still well defined. However, the conformal perturbation theory cannot be applied literally in this case. More generally, at the moment, it is not entirely clear how one can calculate the LHS of (3.15) for arbitrary values of a_i and k_i in the SS -model. In particular, as was mentioned earlier, the knowledge of the exact S -matrix is not of much help in solving this problem. Therefore, as a first step in proving the correspondence (3.15), it would be desirable to derive the UV and IR asymptotics of $E_{\mathbf{k}}^{(\text{vac})}$, discussed above, from the differential equation side. Fortunately, this could be done analytically by using an auxiliary linear problem associated with the shG equation (3.8). The derivation is rather technical and will be published elsewhere. Here we only present the results of our numerical work in support of the conjecture (3.15). The shG equation has been solved numerically for various sets of the parameters a_i and k_i . We found that the resulting values of the RHS of (3.15) are in a good agreement with the UV and IR asymptotic formulae (2.6) and (2.13). Some (small) part of the available numerical data is presented in Fig. 4 and Table 1.

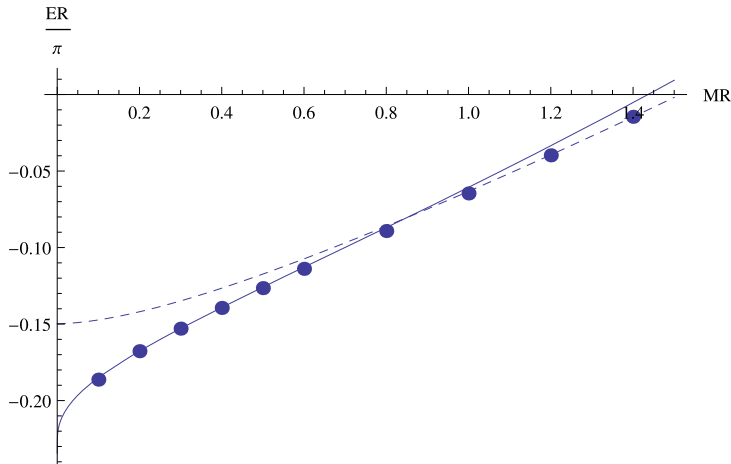


Fig. 4. Numerical values of the dimensionless k -vacuum energy $\frac{R}{\pi} E_{\mathbf{k}}^{(\text{vac})}$ versus the variable $r = MR$ for $a_1 = 1.7$, $a_2 = 1.5$, $k_1 = \frac{4}{17} = 0.235 \dots$, $k_2 = \frac{1}{3}$. The solid and dashed lines represent the small- R , (2.6), and large- R asymptotics (2.13), respectively. The heavy dots represent the LHS of (3.15) calculated from numerical solutions of the shG equation. The corresponding numerical values are presented in Table 1.

Table 1

The dimensionless k -vacuum energy $\frac{R}{\pi} E_{\mathbf{k}}^{(\text{vac})}$ as a function of the variable MR .

MR	r.h.s of Eq. (3.15)	Eq. (2.6)	Eq. (2.13)
0.1	−0.18631	−0.18510	−0.14729
0.2	−0.16773	−0.16770	−0.14197
0.3	−0.15230	−0.15288	−0.13487
0.4	−0.13945	−0.13913	−0.12654
0.5	−0.12651	−0.12589	−0.11731
0.6	−0.11390	−0.11288	−0.10739
0.8	−0.08919	−0.08695	−0.08605
1.0	−0.06463	−0.06061	−0.06327
1.2	−0.03973	−0.03346	−0.03941
1.4	−0.01456	−0.00524	−0.01461

4. Conclusion

In this work we propose the formula (3.15) for the k -vacuum energy in the SS -model for arbitrary values of parameters a_1 and a_2 . Our conjecture is in agreement with expected UV and IR asymptotics of $E_{\mathbf{k}}^{(\text{vac})}$. This can be shown analytically by using the auxiliary linear problem associated with the shG equation. The derivation exploits standard techniques of the inverse scattering method and we did not present it in this short note. Instead, we reported a few numerical verifications supporting our conjecture. Another analytical result, which also remained beyond the scope of the paper, is a system of integral equations for the calculation of $E_{\mathbf{k}}^{(\text{vac})}$. We are planning to discuss it in a separate publication.

Finally, note that our proposal for the k -vacuum energy can be extended to the whole energy spectrum of the SS -model, following the approach of Ref. [15] previously applied to the symmetric regime of the Fateev model.

As for future work, we believe that the most intriguing generalization lies in study of a remarkable two-parameter deformation of the general principal chiral field discovered by Klimčik [16,17]. In Ref. [18] it was pointed out that in the simplest $SU(2)$ case the Klimčik NLSM coincides with the SS -model.

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